We physicists know the importance of dimensional analysis. Sometimes it is possible to write down the correct expression for a physical quantity (up to dimensionless constants) using only dimensional analysis, without resorting to complicated math. Another important tool for physicists is order-of-magnitude estimation, also known as Fermi estimation. When considering whether a question is worth turning into a research project, it is often a good idea to do a quick, back-of-the-envelope calculation before diving in. Several of the problems below are adapted from *Gravitational waves in Physics and Astrophysics, an artisan's guide* (Miller, Yunes).

1. We often work in geometric units where $G = c = 1$, but when it is time to do calculations, we need to relate our expressions back to physical units. One could reintroduce G's and c's into our expressions, then plug in parameter values in SI units. However, it is usually easier to leave our expressions in geometric units and instead express all quantities in terms of a single unit (time, length, or mass).

   (a) Express 1 km and 1 kg in terms of seconds.

   (b) What about 1 $M_\odot$ or 1 Mpc?

   (c) What would you do if you wanted to express all quantities in terms of km? What about kg?

   (Note: $c = 2.99792458 \times 10^5$ km s^{-1}, $G = 6.67430 \times 10^{-20}$ km^3 kg^{-1} s^{-2}, 1 $M_\odot = 1.988416 \times 10^{30}$ kg, 1 Mpc $= 3.086110^{19}$ km)

(a) $[c] = m\,s^{-1}$ $\quad$ $[G] = Km^{-3}\,Kg^{-1}\,s^{-2}$

$Km\,[c]^{-1} = s$

$1Km = 1Km\,(2.99 \times 10^5\, m\, s^{-1})^{-1} = 3.34 \times 10^{-6}\, s$

$Kg\,[G]\,[c]^{-3} = s$

$1Kg = 1Kg\,(6.67 \times 10^{-20}\, Km^3 Kg^{-1} s)\,(2.99 \times 10^5\, m\, s^{-1})^{-3} = 2.48 \times 10^{-36}\, s$

(b) $1\,M_\odot = 1.99 \times 10^{30}\, Kg = 4.9 \times 10^{-6}\, s$

$1\,Mpc = 3.09 \times 10^{19}\, Km = 1.03 \times 10^{14}\, s$

(c) to convert to m multiply by c

to convert to Kg multiply by $G^{-1} c^3$

2. How far (in Mpc) can a solar mass black hole binary be from Earth and still be detectable at our current ground-based observatories? Assume that the detectors are sensitive to strains of approximately $h \sim 10^{-23}$ in the frequency range $f \sim 10 - 10^3$ Hz. Recall that the dominant term in the gravitational wave metric perturbation is

$$h^{ij} \sim \frac{1}{d} \ddot{I}^{<ij>} \, .$$

$I \sim m r^2 \qquad \frac{d}{dt} \sim \omega$

$h \sim \frac{1}{d} \omega^2 m r^2$

Kepler's Law $\quad r^3 = \frac{m}{\omega^2}$

$h \sim \frac{1}{d} \omega^{2/3} m^{5/3}$

$d = h^{-1} \omega^{2/3} m^{5/3}$

$m \sim 10 \, M_\odot = 5 \times 10^{-5} \, s$

$d = 10^{23} \left(10^3 s^{-1}\right)^{2/3} \left(10^{-5} s\right)^{5/3} = 10^{50/3} s \sim 10^{17} s$

$d = 10^{17} s \left(10^{14} s \, Mpc^{-1}\right)^{-1} = 10^3 \, Mpc$

3. Could LIGO detect the gravitational waves produced by a person waving their arms with a frequency of 100 Hz at a distance of 10^4 km? How close would they have to be? (Based on Exercise 2, Chapter 1)

$$h \sim \frac{1}{d} \omega^2 r^2 m$$

$$d = 10^4 \times 3.3 \times 10^{-6}\,s = 3.3 \times 10^{-2}\,s$$

$$\omega = 10^2\,s^{-1}$$

$$r = 10^{-3} \times 3.3 \times 10^{-6}\,s = 3.3 \times 10^{-9}\,s$$

$$m = 10 \times 2.5 \times 10^{-36}\,s = 2.5 \times 10^{-35}\,s$$

$$h \sim (10^{-2})^{-1} (10^{-2})^2 (10^{-9})^2 (10^{-35}) = 10^{-55}$$

$$\boxed{10^{-55} \ll 10^{-22}}$$

LIGO cannot detect

$$d = 10^{22} (10^{-2})^2 (10^{-9})^2 (10^{-35})$$

$$= 10^{-35}\,s$$

$$= 10^{-35} (10^6)\,s = 10^{-29}\,\text{km} \,!$$

4. As binary systems radiate energy through graviational waves, the total energy of the binary decreases and the orbital separation shrinks. Do we need to be worried about the Earth inspiralling into the sun due to the energy lost to gravitational radiation? Note that the dominant term of the power emitted by gravitational radiation is

$$\mathcal{P} \sim \dddot{I}^{<ij>} \dddot{I}_{<ij>} .$$

(Based on Exercise 5, Chapter 1)

$$P \sim \dddot{Q}^{ij} \dddot{Q}_{ij} + \ldots \sim \omega^6 (m r^2)^2 = (\omega m)^{10/3}$$

$$\Delta E \sim \frac{m^2}{r} = m^{5/3} \omega^{2/3}$$

$$P = -\frac{\Delta E}{\Delta t}$$

$$\Delta t = \frac{\Delta E}{P} = m^{-5/3} \omega^{-8/3}$$

$$m = 1 \, M_{sol} = 10^{-6} \, s$$

$$\omega = 2\pi \, (60 \times 60 \times 24 \times 365)^{-1} \sim 10^{-7} \, s^{-1}$$

$$\Delta t = (10^{-6})^{-5/3} (10^{-7})^{-8/3} \, s = 10^{86/3} \, s = 10^{29} \, s$$

$$= 10^{22} \text{ years}$$

we left out factors of the mass ratio $q = 10^{-6}$.
does that matter?

5. We typically say that a binary transitions from inspiral to merger once the orbital separation reaches the innermost stable circular orbit (ISCO), which occurs at $r = 6m$. Use Kepler's law $\omega^2 = mr^{-3}$ and the fact that the gravitational frequency is $f = \omega/\pi$ to determine the gravitational wave frequency at the ISCO in Hz as a function of ($m/$ $M_\odot$). Do this calculation exactly (no Fermi estimates). For what range of masses will binaries merge in the LIGO band?

$$r = m^{1/3}\,\omega^{-2/3} = m^{1/3} f^{-2/3}\,\pi^{-2/3}$$

$$f = \left[6m\left(m^{-1/3}\,\pi^{2/3}\right)\right]^{-3/2}$$

$$= \frac{1}{6\sqrt{6}\,\pi}\,m^{-1}$$

$$= \frac{1}{6\sqrt{6}\,\pi}\,\frac{1}{M_\odot}\left(\frac{M_\odot}{m}\right) = 4.4 \times 10^{3}\ \mathrm{Hz}\left(\frac{M_\odot}{m}\right)$$

$m = 1 - 100\ M_\odot$ will merge in the LIGO band

6. The Hulse-Taylor pulsar is in a binary system with an orbital period of 7.75 hours. The binary is 6.4×10^{-3} Mpc from earth. The pulsar and its companion each have a mass of approximately 1.4 $M_\odot$. Is the gravitational wave signal produced by this system currently detectable in either the LIGO or the LISA band? LISA will be sensitive to strains of approximately $h \sim 10^{-20}$ in the frequency range $f \sim 10^{-4} - 1$ Hz. If it is not currently detectable, will it ever be detectable? How far in the future will it become detectable? Actually compute an integral to answer the last question. Use the the ballance equation

$$\mathcal{P} = -\frac{dE}{dt} = -\frac{dE}{d\omega}\frac{d\omega}{dt}\,. \tag{1}$$

$\omega = 2\pi\,(7.75 \times 60 \times 60)^{-1}\ s^{-1} \sim 2 \times 10^{-4}\ s^{-1}$

$d = 6.4 \times 10^{-3}\,(10^{14})\ s = 6 \times 10^{11}\ s$

$m = 3\,(5 \times 10^{-6}\ s) = 1.5 \times 10^{-5}\ s$

$h = \frac{1}{d}\,\omega^{2/3}\,m^{5/3} = 5 \times 10^{-23}$

LISA - no, good frequency, but strain is too low

When it Leaves the LISA band $\omega \sim 1$, $h \sim 10^{-20}$
maybe detectable, probably not

LIGO - no, good strain, but frequency is too low
When it enters the LIGO band $\omega \sim 10$, $h \sim 7 \times 10^{-20}$
detectable! but how long will it take to get there?

$E = -m^{5/3}\,\omega^{2/3}$ $\qquad P = (\omega m)^{10/3}$

$P = -\frac{dE}{dt} = \frac{dE}{d\omega}\frac{d\omega}{dt}$

$\Delta t = \int_{\omega_1}^{\omega_2} \frac{2}{3} m^{-5/3}\,\omega^{-1/3}\,(m\omega)^{-10/3}\ d\omega$

$= -\frac{1}{4}\,m^{-5/3}\,\omega^{-8/3}\Big|_{\omega=2\times10^{-4}}^{\omega=10} = 2 \times 10^{17}\ s = 7 \times 10^{9}$ years!