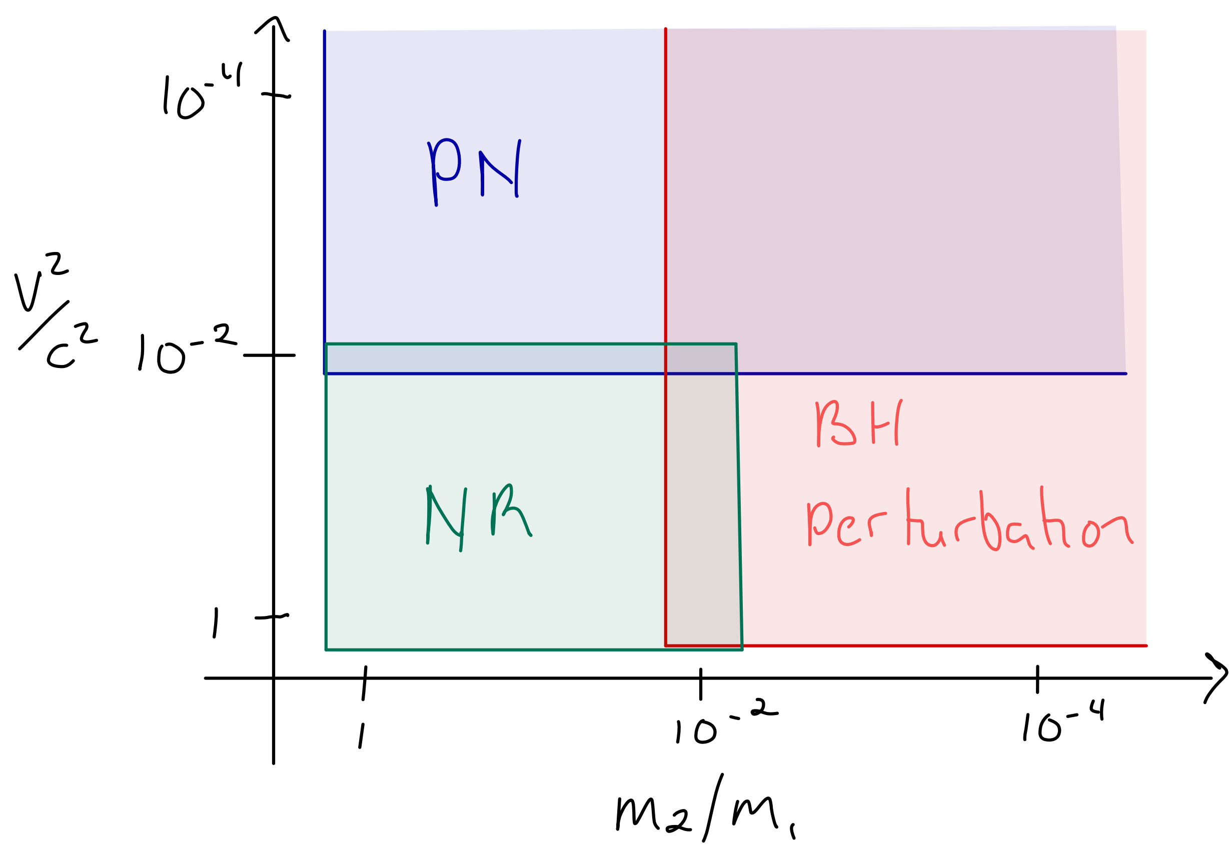
Stationary Phase Approximation (SPA)

post-Newtonian theory: slow: $\left(\frac{v}{c}\right)^2 \ll 1$ weak fields: $\frac{Gm}{rc^2} \ll 1$

circular binary $\Rightarrow$ relate r, v via ω

$$\left.\begin{aligned} \omega^2 &= \frac{Gm}{r^3} + \dots \\ v &= \omega r \end{aligned}\right\} \quad x \equiv (Gm\omega)^{2/3} c^{-2} \quad \left\}\begin{aligned} \omega &= \frac{c^3}{Gm} x^{3/2} \\ r &= \frac{Gm}{c^2 x} + \dots \\ v^2/c^2 &= x + \dots \end{aligned}\right.$$

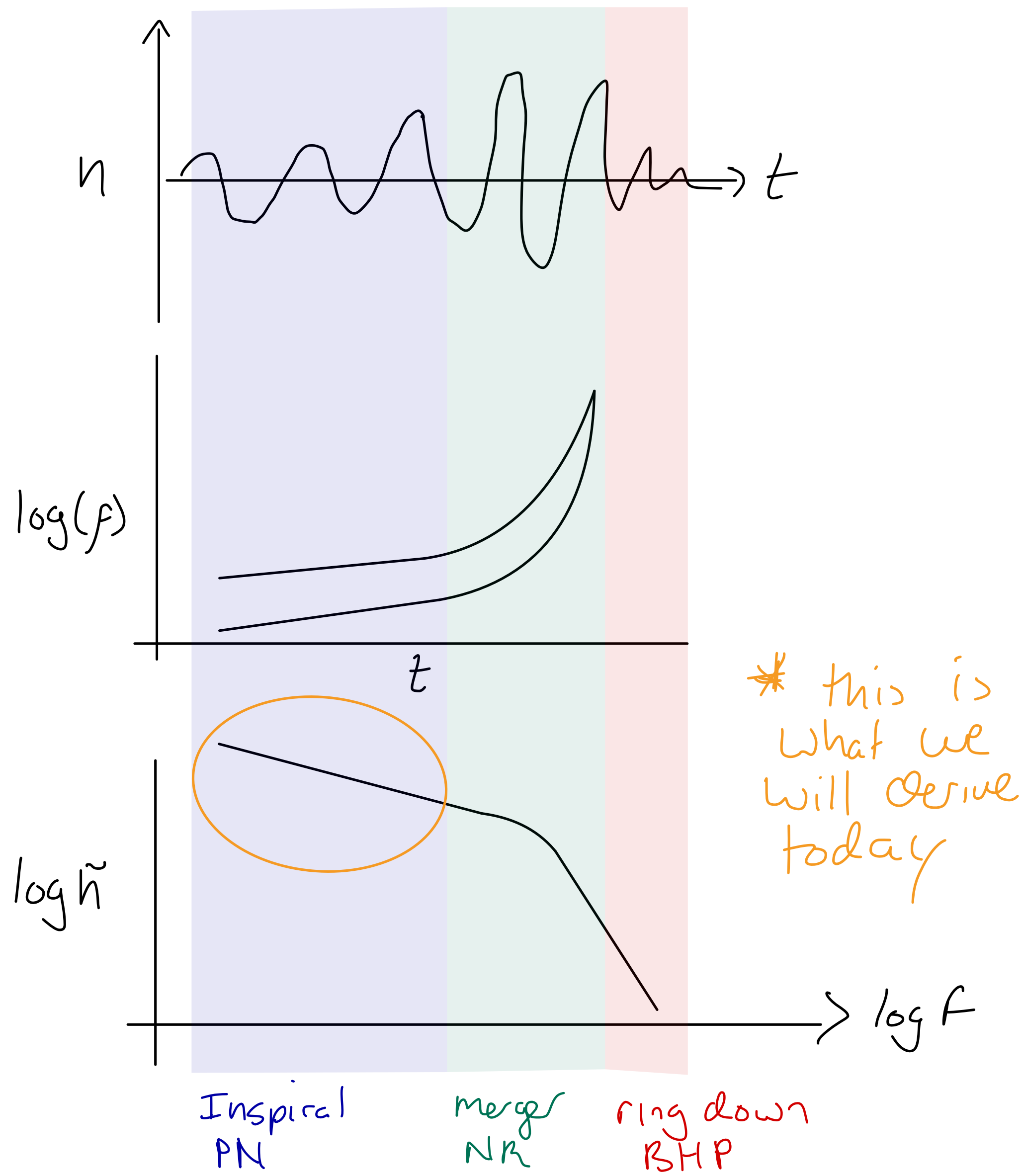


set $G = c = 1$ from now on...

We need $\tilde{h}(f)$

1. signal identification (matched filtering)
2. parameter estimation

What does a gravitational wave signal look like?

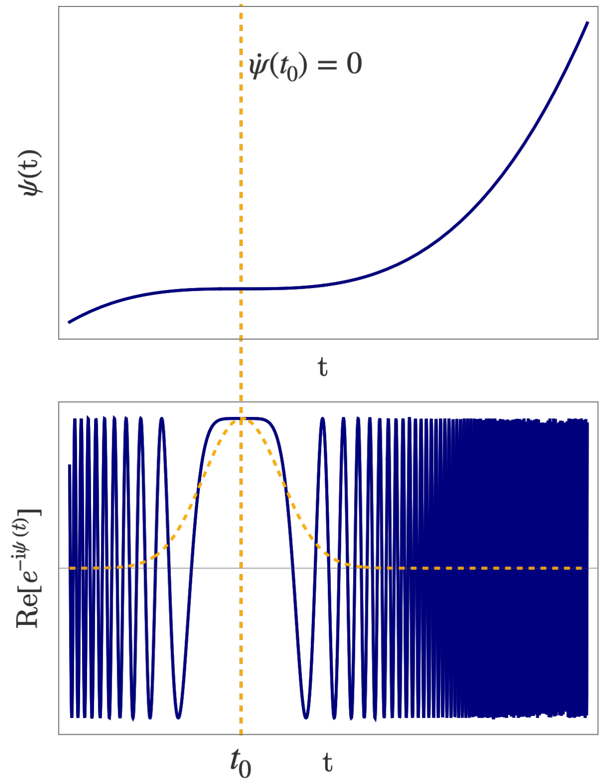


SPA ⇒ directly compute $\tilde{h}(f)$
when radiation-reaction time scale ≪ orbital period

what do we need to apply it ?

1. Amplitude varies slower than phase
2. orbital frequency / phase are posative and monotonically increasing

Resources: 0906.0313, SPA wikepedia, Maggiore, 0907.0700



First use SPA to compute the integral of a generic oscillating function

$$F(t) = A(t)\, e^{-i\psi(t)} \qquad I = \int F(t)\, dt$$

If $F(t)$ has a "stationary point" t_0 where $\psi(t_0) = 0$ the area near this point will contribute most to the integral

Expand $A(t)$, $\psi(t)$ about t_0

$$\psi(t) = \psi(t_0) + \cancelto{0}{\dot{\psi}(t_0)}(t-t_0) + \tfrac{1}{2}\ddot{\psi}(t_0)(t-t_0)^2 + \dots$$

$$A(t) = A(t_0) + \dots \qquad \text{(Amplitude varies slowly)}$$

$$I = A(t_0)\, e^{-i\psi(t_0)} \int e^{-\frac{i}{2}\ddot{\psi}(t_0)(t-t_0)^2} dt \qquad \text{Gaussian!}$$

$$I = A(t_0) \left[\frac{2\pi}{\ddot{\psi}(t_0)}\right]^{1/2} e^{-i[\psi(t_0) + \pi/4]}$$

Now lets apply the SPA to obtain the frequency domain GW Strain

$$h^{ij} = h_+(t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + h_\times(t)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\iota = 0$

$$h_+ = \tfrac{1}{2}\cancelto{1}{(1+\cos^2\iota)}\, A(t)\cos[2\phi(t)] \qquad h_\times = \cancelto{1}{\cos\iota}\, A(t)\sin[2\phi(t)]$$

$\phi(t)$ = orbital phase

$m = m_1 + m_2$

$r = |\vec{r}_1 - \vec{r}_2|$

$\eta = \dfrac{m_1 m_2}{m^2}$ $\qquad m\eta = \mu$

$v = |\vec{v}_1 - \vec{v}_2|$

$$h^{ij} = \frac{1}{d}\ddot{I}^{ij} + \ldots \sim \frac{1}{d}\omega^2 m r^2$$

$$A(t) = -4\frac{1}{d}\eta m \omega^2 r^2 + \ldots$$

$$A(t) = \boxed{-4\frac{1}{d}\eta m x + \ldots}$$

While we are at it,

$$E = \tfrac{1}{2} m\eta v^2 - \frac{m^2\eta}{r} = \boxed{-\tfrac{1}{2} m\eta x + \ldots}$$

total energy

$$P \sim \dddot{I}^{ij}\dddot{I}_{ij} + \ldots \sim \omega^6 (m r^2)^2$$

$$= \frac{32}{5}\eta^2 m^2 \omega^6 r^4 = \boxed{\frac{32}{5}\eta^2 x^5 + \ldots}$$

Power radiated by GWs

$$h(t) = h_+ - i h_\times = A e^{-2i\phi}$$

Collect $h_+ / h_\times$ into a single term for simplicity

$$\tilde{h}(f) = \int h(t) e^{2\pi i f t} \, dt$$

$$= \int A(t) e^{2\pi i f t} e^{-2i\phi(t)} \, dt$$

We can write $\tilde{h}(f) = \tilde{A}(f) e^{-i\tilde{\Phi}(f)}$ in terms of an Amplitude and phase:

$$\tilde{A}(f) = A(t_0) \left[\frac{2\pi}{\ddot{\Psi}(t_0)} \right]^{1/2} \qquad \tilde{\Phi}(f) = \Psi(t_0) + \pi/4$$

the total phase of the integrand is

$$\Psi(t) = -2 \left[\pi f t - \phi(t) \right]$$

← this carries a minus sign because of the way we defined F(t) on the first page

$$\dot{\Psi}(t) = -2 \left[\pi f - \dot{\phi}(t) \right]$$

$$\ddot{\Psi}(t) = 2 \ddot{\phi}(t) = 2 \dot{\omega}(t)$$

stationary point: $\dot{\Psi}(t_0) = 0 \implies \dot{\phi}(t_0) = \pi f \equiv \omega(t_0)$

the Amplitude and phase of $\tilde{h}(f)$ are

$$\boxed{\tilde{A}(f) = A(t_0) \left(\frac{\pi}{\dot{\omega}(t_0)} \right)^{1/2}}$$

= we need to solve for these

$$\tilde{\Phi}(f) = 2 t_0 \omega(t_0) - 2\phi(t_0) - \pi/4$$

$$X_0 = X(t_0) = \left[m \, \omega(t_0) \right]^{2/3}$$

$$= (\pi m f)^{2/3}$$

$$\boxed{\tilde{\Phi}(f) = -2 t_0 \frac{X_0^{3/2}}{m} + 2\phi(t_0) + \pi/4}$$

Solve for $\tilde{\Phi}(f)$ First

Ballance equation $\quad P = -\frac{dE}{dt} = -\frac{dE}{dx}\frac{dx}{dt} = -E'\frac{dx}{dt}$

Not Newtonian! binary loses energy to GW

Find t_0:

$$\frac{E'}{P} = -\frac{5}{64}\frac{m}{\eta}x^{-5}$$

$$dt = -\frac{E'}{P}dx$$

$$t_0 = t_c - \int \frac{E'}{P}dx \Big|_{x=x_0}$$

$$t_c + \frac{5}{64}\frac{m}{\eta}\int x^{-5}dx \Big|_{x=x_0}$$

$$\boxed{t_0 = t_c - \frac{5}{256}\frac{m}{\eta}x_0^{-4}}$$

Find $\phi(t_0)$

$$\omega = \frac{d\phi}{dt} = \frac{d\phi}{dx}\frac{dx}{dt} = \underline{-\frac{d\phi}{dx}\frac{P'}{E} = \frac{x^{3/2}}{m}}$$

$$d\phi = -\frac{E'}{P}\frac{x^{3/2}}{m}dx$$

$$\phi(t_0) = \phi_c - \int \frac{E'}{P}\frac{x^{3/2}}{m}dx \Big|_{x=x_0}$$

$$= \phi_c + \frac{5}{64}\frac{1}{\eta}\int x^{-7/2}dx \Big|_{x=x_0}$$

$$\boxed{\phi(t_0) = \phi_c - \frac{1}{32}\frac{1}{\eta}x_0^{-5/2}}$$

$$\tilde{\phi} = -2t_0 \frac{x_0^{3/2}}{m} + 2\phi(t_0) - \pi/4$$

$$= -2\left(t_c - \frac{5}{256}\frac{m}{\eta} x_0^{-4}\right)\frac{x_0^{3/2}}{m} + 2\left(\phi_c - \frac{1}{32}\frac{1}{\eta} x_0^{-5/2}\right) - \pi/4$$

$$= -2t_c \frac{x_0^{3/2}}{m} + 2\phi_c - \frac{3}{128}\frac{1}{\eta} x_0^{-5/2} + \pi/4$$

$$\tilde{\phi}(f) = -2\pi f t_c + 2\phi_c + \pi/4 - \frac{3}{128}\frac{1}{\eta}(\pi m f)^{-5/3}$$

$$\boxed{\mathcal{M} = m\eta^{3/5}}$$

chirp mass!

$$\boxed{\tilde{\phi}(f) = -2\pi f t_c + 2\phi_c + \pi/4 - \frac{3}{128}\mathcal{M}^{-5/3}(\pi f)^{-5/3}}$$

Next, Solve for $\tilde{A}(f)$

Find $\dot{\omega}(t_0)$:

$$\dot{\omega} = \frac{d\omega}{dx}\frac{dx}{dt}$$

$$\dot{\omega} = \frac{3}{2}\frac{1}{m} x^{1/2}\frac{dx}{dt}$$

$$\dot{\omega}(t_0) = \frac{3}{2}\frac{1}{m} x^{1/2} \left.\frac{P}{E'}\right|_{x = x_0}$$

$$\boxed{\dot{\omega}(t_0) = \frac{96}{5}\frac{\eta}{m^2} x_0^{11/2}}$$

Find $A(t_0)$:

$$\boxed{A(t_0) = -4\frac{1}{d}\eta m x_0}$$

$$\tilde{A}(f) = A(t_0)\left(\frac{\pi}{\dot{\omega}(t_0)}\right)^{1/2}$$

$$= -\left(\frac{5\pi}{6}\right)^{1/2}\frac{1}{d} m^2 \eta^{1/2} x^{-7/4}$$

$$\tilde{A}(f) = -\left(\frac{5\pi}{6}\right)^{1/2}\frac{1}{d} m^2 \eta^{1/3} (\pi m f)^{-7/6}$$

$$\boxed{\tilde{A}(f) = -\left(\frac{5\pi}{6}\right)^{1/2}\frac{1}{d}\mathcal{M}^{5/6}(\pi f)^{-7/6}}$$

Note: $\tilde{h}_+ = \frac{1}{2}\tilde{h}$, $\tilde{h}_\times = \frac{i}{2}h$ $\Rightarrow$ $\tilde{h} = \tilde{h}_+ - i\tilde{h}_\times$

How to extend calculation?

must add corrections to:

Keplers law $[\omega(r)]$, $A(t)$, $E(x)$, $P(x)$

What we have computed is the Newtonian term of the TaylorF2 waveform

see 2012.0135 for an overview of different waveform approximants