Post-Newtonian Binaries

Here I outline how to compute the ingredients we needed to compute the frequency domain waveform earlier this week.

I do most calculations at Newtonian order and then write down the 1PN corrections.

The main PN references are: Gravity - Poisson + Will
Luc Blanchet's Living Review (1310.1528)

Einstein's Equation:

$$G^{\mu\nu} = 8\pi \frac{G}{c^4} T^{\mu\nu}$$

Linear wave equation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h_{\mu\nu} << 1$$

$\Rightarrow$ discard $h_{\mu\nu}^2$
$\Rightarrow$ trace reversed: $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$
$\Rightarrow$ harmonic gauge. $\partial_\mu \bar{h}^{\mu\nu} = 0$

$$\Box \bar{h}^{\mu\nu} = -16\pi \frac{G}{c^4} T^{\mu\nu}$$

this does not include non linear effects (of course)
but gravitational waves are non linear! Waves make more waves

Relaxed wave equation (P+W ch 6)

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \qquad h^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g}\, g^{\alpha\beta} \qquad g = \det(g_{\mu\nu})$$

$\Rightarrow$ don't discard any terms

$\Rightarrow$ $\partial_\mu h^{\mu\nu} = 0$

$\Rightarrow$ Collect non-linear terms $\Lambda^{\mu\nu} = \Lambda^{\mu\nu}(h^{\mu\nu}, \partial_\alpha h^{\mu\nu})$

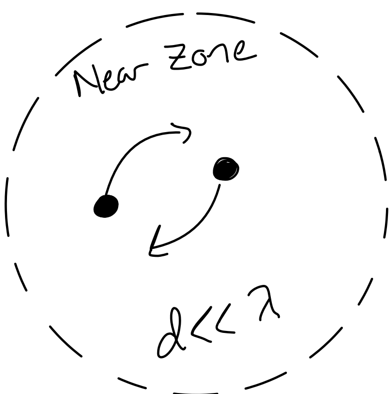
$$\Box h^{\mu\nu} = -16\pi \frac{G}{c^4} \tau^{\mu\nu} \qquad \tau^{\mu\nu} = \sqrt{-g}\left(T^{\mu\nu} + \Lambda^{\mu\nu}\right)$$

This equation is exact but reduces to the linear equation

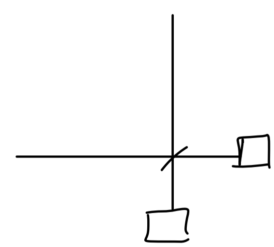
We will solve this iteratively in the post-Minkowski approximation

Assume weak fields: $Gm/rc^2 \ll 1$ $\quad h^{\mu\nu} = G h_1^{\mu\nu} + G^2 h_2^{\mu\nu} + \ldots$

Solve in 2 regions: P+W Ch. 6



Wave Zone: $d \gg \lambda_c$



$$h_{00} = \frac{2}{c^2} U + \frac{2}{c^4}\left(\psi - U^2\right) + \mathcal{O}(c^{-6})$$

$$h_{i0} = -\frac{4}{c^3} U_i + \mathcal{O}(c^{-5})$$

$$h_{ij} = \frac{2}{c^2} U \,\delta_{ij} + \mathcal{O}(c^{-4})$$

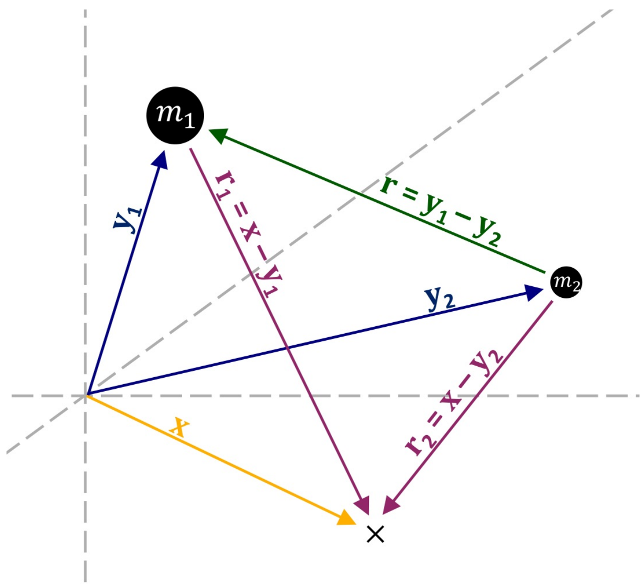
$$\boxed{h^{ij} = \frac{2G}{c^4 d}\left[\ddot{I}^{jk} + \left(\begin{array}{l}\text{higher}\\ \text{order}\\ \text{multipoles}\end{array}\right)\right]}$$

For a perfect fluid: mass density ρ, velocity field v^i, pressure p, energy density ε

$$\nabla^2 U = -4\pi G \rho$$
$$\nabla^2 U^i = -4\pi G v^i$$
$$\nabla^2 \psi = -4\pi G \rho\left(3/2 - U + \varepsilon/\rho + 3p/\rho\right) + \partial_t^2 U$$

See P+W Eq 8.2
Section 9.6

We are going to consider a binary system. First we will establish some conventions



For body A $(A = 1, 2)$:

$$\vec{v}_A = \frac{d\vec{y}_A}{dt} \qquad \vec{a}_A = \frac{d\vec{v}_A}{dt}$$

$$\vec{r} = \vec{y}_1 - y_2 = \vec{r}_2 - \vec{r}_1$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{v}_1 - \vec{v}_2 \qquad \vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_1 - \vec{a}_2$$

unit vectors $\vec{n}_A = \frac{\vec{r}_A}{r} \qquad \vec{n} = \frac{\vec{r}}{r}$

indices $\alpha, \beta, \gamma \ldots$ space-time components
Indices i, j, k spatial components

$\vec{r}$ = vector r^i = components r = magnitude

$V^\alpha = [c, \vec{v}]$ unnormalized 4-velocity

$u^\alpha = \gamma(v^\alpha)[c, \vec{v}]$ normalized 4-velocity

$$\gamma(v^\alpha) = \left[-V^\alpha V^\beta g_{\alpha\beta}/c^2\right]^{1/2}$$

$\left[\text{reduces to } \gamma = \left[1 - \frac{v^2}{c^2}\right]^{-1/2} \text{ in flat space}\right]$

$$\dot{r} = \frac{dr}{dt} = \vec{v} \cdot \vec{n}$$

these are the conventions from Blanchet.
P+W uses $\vec{y}_A \Rightarrow \vec{r}_A$ and $\vec{r}_A \Rightarrow \vec{s}_A$

We will begin in the near zone to obtain

- EoM: $\vec{a} = \vec{a}(\vec{r}, \vec{v})$
- Energy E
- Kepler's law $\omega^2 = \frac{GM}{r^3} + \ldots$,

$$\rho = m_1 \delta(\vec{x} - \vec{y}_1) + m_2 \delta(\vec{x} - \vec{y}_2)$$

$$U = \frac{Gm_1}{r_1} + (1 \leftrightarrow 2)$$

$$U^i = \frac{Gm_1}{r_1} v_1 + (1 \leftrightarrow 2)$$

$$\psi = \frac{Gm_1}{r_1}\left[2v_1^2 - \tfrac{1}{2}(\vec{n}_1 \cdot \vec{v}_1)^2 - \frac{Gm_2}{r}\left(1 - \frac{n \cdot r_1}{2r}\right)\right] + (1 \leftrightarrow 2)$$

First we will get $\vec{a}_1 \Rightarrow a_2$ by symmetry $\Rightarrow \vec{a} = \vec{a}_1 - \vec{a}_2$

the path of body 1 will minimize proper time

$$d\tau_1 = -c^{-2} g_{\alpha\beta}\, dy_1\, dy_1$$

the relevant legrangian is

$$L = mc\left[-g_{\alpha\beta}\frac{dy_1^\alpha}{dt}\frac{dy_1^\beta}{dt}\right]^{1/2} \qquad \frac{dy_1^\alpha}{dt} = (c, \vec{v})$$

$$L = mc^2\left[-g_{00} - 2c^{-1} g_{0i} v^i - c^{-2} g_{ik} v^i v^k\right]^{1/2}$$

If we want the EoM to c^{-2}, we need

$$\begin{aligned} g_{00} &\sim \mathcal{O}(c^{-4}) \\ g_{i0} &\sim \mathcal{O}(c^{-3}) \\ g_{jk} &\sim \mathcal{O}(c^{-2}) \end{aligned}$$

The EoM is the usual geodesic Equation

$$\frac{d^2 y_1^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{d y_1^\alpha}{d\tau^2} \frac{d y_1^\alpha}{d\tau^2}$$

Write in terms of coordinate time using $\frac{d y_1^\alpha}{d\tau} = \frac{d y_1^\alpha}{dt} \frac{dt}{d\tau}$

and the 0th component of the EoM to get to eliminate $\frac{d^2 t/d\tau^2}{(dt/d\tau)^2}$

$$a_1^i = -\left(\Gamma^i_{\beta\gamma} - \frac{v_1^i}{c} \Gamma_{\beta\gamma}\right) v_1^\beta v_1^\gamma$$

remember, $v_1^\alpha = (c, \vec{v}_1)$ so

$$a_1^i = -c^2 \underbrace{\Gamma^i_{00}}_{-\frac{1}{c^2}\partial_i U} + \ldots$$

Christoffels from P+W Eq. 8.15

$$= \partial_i U + \ldots$$

$$= \partial_i \left[\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right] + \ldots$$

$$= -\left(\frac{Gm_1}{r_1^2} n_1^i + \frac{Gm_2}{r_2^2} n_2^i\right)$$

$r_A = \vec{x} - \vec{y}_A$ we need to evaluate this at $\vec{x} = \vec{y}_1$

the first term blows up but this is just a result of our point particle approximation so set it to zero (Hadamard Regularization)

$$\vec{a}_1 = -\frac{Gm_2}{r^2} \vec{n} \quad \text{as expected}$$

$$\vec{a} = -\frac{Gm}{r^2}\vec{n} - \frac{Gm}{r^2c^2}\left\{\left[(1+3\eta)v^2 - \frac{3}{2}\dot{r}^2 - 2(2+\eta)\frac{Gm}{r}\right]\vec{n} - 2(2-\eta)\dot{r}\vec{v}\right\}$$

Following P+W chapt 10

We can determine E by enforcing $\frac{dE}{dt} = 0$

$$E = \underbrace{m\eta\left(\tfrac{1}{2}v^2 - \frac{Gm}{r}\right)}_{\text{usual Newtonian result}} + \underbrace{c^{-2}\left[A_1 v^4 + A_2 v^2 \dot{r}^2 + A_3 \dot{r}^4 + A_4 v^2 \frac{Gm}{r} + A_5 \dot{r}^2 \frac{Gm}{r} + A_6\left(\frac{Gm}{r}\right)^2\right]}_{\text{most general ansatz given form of } \vec{a}}$$

Determine A_i by setting $\frac{dE}{dt} = 0$ using $\frac{d\vec{r}}{dt} = \vec{v}$ and $\frac{d\vec{v}}{dt} = \vec{a}$ above

I am just going to show that this is true at c^0

$$\frac{d}{dt}v^2 = \frac{d}{dt}(\vec{v}\cdot\vec{v}) \qquad\qquad \frac{dr^{-1}}{dt} = -r^{-2}\dot{r}$$

$$= 2\vec{v}\cdot\vec{a}$$

$$= -2\frac{Gm}{r^2}\dot{r} + \mathcal{O}(c^{-2})$$

$$\frac{dE}{dt} = m\eta\left[\tfrac{1}{2}\left(-2\frac{Gm}{r^2}\dot{r}\right) - Gm\left(-r^{-2}\dot{r}\right)\right] + \mathcal{O}(c^{-2}) = 0 + \mathcal{O}(c^{-2})$$

Writing down E to $\mathcal{O}(c^{-2})$ [P+W Eq 10.3]

$$E = m\eta\left(\tfrac{1}{2}v^2 - \frac{Gm}{r}\right) + m\eta c^{-2}\left\{\tfrac{3}{8}(1-3\eta)v^4 + \frac{Gm}{r}\left[(3+\eta)v^2 + \eta\dot{r} + \frac{Gm}{r}\right]\right\} + \mathcal{O}(c^{-4})$$

Now we want to correct Keplers laws. this will require us to show that orbital motion occurs in a plane so we need $\vec{L}$

$$\vec{L} = \underbrace{m\eta(\vec{r}\times\vec{v})}_{\text{Newtonian}} + c^{-2}\Big[(\vec{r}\times\vec{v})\Big(A_1 v^2 + A_2 \dot{r}^2 + A_3 \frac{Gm}{r}\Big)$$
$$\underbrace{+\,\vec{r}\,\Big(\quad " \quad\Big) + \vec{v}\,\Big(\quad " \quad\Big)\Big]}_{\text{ansatz}}$$

set $\frac{d}{dt}\vec{L} = 0$

Again, I will do this to $\mathcal{O}(c^0)$ and write down $\mathcal{O}(c^{-2})$

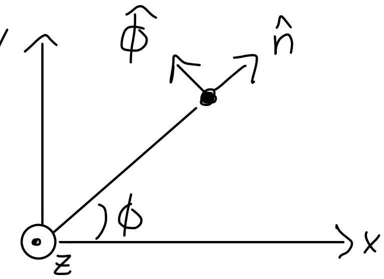
$$\frac{d}{dt}(\vec{r}\times\vec{v}) = \cancelto{0}{\vec{v}\times\vec{v}} + \vec{r}\times\vec{a}$$
$$= \frac{Gm}{r^3}\,\vec{r}\times\vec{r} + \mathcal{O}(c^{-2}) = 0 + \mathcal{O}(c^{-2})$$

$$\vec{L} = m\eta(\vec{r}\times\vec{v})\Big\{1 + c^{-2}\Big[\tfrac{1}{2}(1-3\eta)v^2 + (3+\eta)\frac{Gm}{r}\Big]\Big\} + \mathcal{O}(c^{-4})$$

[P+W 10.4]

$\vec{r}\times\vec{v}$ is not conserved, but the direction of $\vec{r}\times\vec{v}$ is

orbital motion occurs in a plane



work in a more suitable basis.

$\hat{n} = [\cos\phi, \sin\phi, 0]$, $\hat{\phi} = [-\sin\phi, \cos\phi, 0]$, $\hat{z} = [0,0,1]$

ϕ is the orbital phase

We can express $\vec{r}, \vec{v}, \vec{a}$ in this basis by simply taking derivatives

$\vec{r} = r\hat{n}$ $\quad\quad$ $\vec{v} = \dot{r}\hat{n} + r\dot{\phi}\hat{\phi}$ $\quad\quad$ $\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{n} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})\hat{\phi}$

write $\vec{a} = (\ldots)\hat{n} + (\ldots)\vec{v}$ as $\vec{a} = (\ldots)\hat{n} + (\ldots)\hat{\phi}$

From this new expression we can read of -

$$\ddot{r} - r\dot{\phi}^2 = -\frac{Gm}{r^2}\left\{1 - c^{-2}\left[(1+3\eta)v^2 - \tfrac{1}{2}(8-\eta)\dot{r}^2 - 2(2+\eta)\frac{Gm}{r}\right]\right\}$$

$$\frac{d}{dt} r^2\dot{\phi} = 2(2-\eta)\frac{Gm}{c^2}\dot{r}\dot{\phi}$$

work in the circular limit $\dot{r} = 0$

$$\dot{\phi} = \text{const} = \omega$$

$$\ddot{r} - r\dot{\phi}^2 = -\frac{Gm}{r^2}\left\{1 - c^{-2}[\ldots]\right\}$$

$$-r\omega^2 = -\frac{Gm}{r^2}\left\{1 - c^{-2}\underline{[\ldots]}\right\}$$

use $v = \omega r$ and $\frac{Gm}{r^3} + \mathcal{O}(c^{-2})$

$$\omega^2 = \frac{Gm}{r^3}\left[1 - c^{-2}(3-\eta)\frac{Gm}{r} + \mathcal{O}(c^{-4})\right]$$

Keplers law!

Now we go to the wave zone to get

- the strain amplitude $A(t)$
- the radiated power P_{GW}

We have $h^{ij} = \frac{2G}{c^4 d} \ddot{I}^{<ij>} + \dots$ P+W Box 7.7

and $P_{GW} = \frac{G}{c^5}\left[\frac{1}{5} \dddot{I}^{<ij>} \dddot{I}^{<ij>} + \dots \right]$ P+W Eq 12.68

Following P+W Sec 12.3.3

We will compute these at Newtonian order so we need $I^{<ij>}$ and its derivatives

$$I^{ij} = \int \rho x^i x^j dx^3 + \dots$$

$$= m_1 y_1^i y_1^j + m_2 y_2^i y_2^j \qquad \text{use } \vec{r} = y_1 - y_2 \text{ and } \vec{r}_1 \vec{m}_1 + \vec{r}_2 m_2 = O(c^2)$$

$$= m\eta r^i r^j$$

take derivatives:

$$\dot{I}^{ij} = m\eta \left(r^i v^j + v^i r^j \right)$$

$$\ddot{I}^{ij} = m\eta \left(2 v^i v^j + r^i a^j + a^i r^j \right)$$

$$= 2m\eta \left(v^i v^j - \frac{Gm}{r^3} r^i r^j \right) \qquad a^i = -\frac{Gm}{r^3} r^i + O(c^{-2})$$

We still have one more derivative to take but I will stop here and STF this because we will need it for h^{ij}.

First... r^i $\quad r n^i \quad v^i = \dot{r} n^i + r\dot{\phi}\phi^i = r\omega\phi^i$

$$\ddot{I}^{ij} = 2m\eta\left(r^2\omega^2\phi^i\phi^j - \frac{Gm}{r}n^i n^j\right) \qquad \frac{Gm}{r} = r^2\omega^2 + \mathcal{O}(c^{-2})$$

$$= 2m\eta r^2\omega^2\left(\phi^i\phi^j - n^i n^j\right)$$

clearly symmetric and $\phi^i\phi_j = 1 = n^i n_j$, so trace free

now we have what we need to compute $h^{ij} = \frac{2G}{c^4 d}\ddot{I}^{<ij>} + \dots$

$$h^{ij} = -4\frac{G}{c^4}\frac{1}{d}m\eta r^2\omega^2\left(n^i n^j - \phi^i\phi^j\right)$$

We can write out the components

$$\left(n^i n^j - \phi^i\phi^j\right) = \begin{pmatrix} \cos 2\phi & \sin 2\phi & 0 \\ \sin 2\phi & -\cos 2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h^{ij} = h_+\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + h_\times\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h_+ = A\cos 2\phi \qquad h_\times = A\sin 2\phi$$

$$A = -\frac{G}{c^4}4\eta\frac{1}{d}m r^2\omega^2 + \dots \qquad x = \left(Gm\omega/c^3\right)^{2/3}$$

$$A = -4\frac{G}{c^2}\frac{\eta m}{d}x\left[1 - x\frac{1}{42}\left(107 - 55\eta\right) + \dots\right]$$

this is not the complete 1PN amplitude. the 1PN term comes from corrections to I^{ij}, but a full 1PN amplitude would contain contributions from higher order multipole moments. this is the amplitude in the PhenomD waveform

Now lets compute $P_{GW} = \frac{1}{5}\frac{G}{c^5}\dddot{I}^{\langle ij\rangle}\dddot{I}^{\langle ij\rangle} + \dots$

$$\ddot{I}^{\langle ij\rangle} = -2\eta m r^2\omega^2\left(n^i n^j - \phi^i\phi^j\right)$$

Note that $\dot{n}^i = -\omega\phi^i \qquad \dot{\phi}^i = \omega n^i$

$$\dddot{I}^{\langle ij\rangle} = 4\eta m\, r^2\omega^3\left[n^i\phi^j + \phi^i n^j\right]$$

We need this square $\left[n^i\phi^j + \phi^i n^j\right] = 2$

$$P_{GW} = \frac{32}{5}\frac{G}{c^5}\eta^2 m^2 r^4\omega^6 + \dots$$

$$P_{GW} = \frac{32}{5}\frac{c^5}{G}\eta^2 x^5\left[1 - x\left(\frac{1247}{336} + \frac{35}{12}\eta\right) + \dots\right]$$